Project systems theory – Solutions

Final exam 2016–2017, Thursday 26 January 2017, 9:00 – 12:00

Problem 1

Consider the model

$$\dot{p}(t) = \Big(2 - \tfrac{1}{6}p(t) - \tfrac{1}{4}q(t)\Big)p(t), \tag{1}$$

$$\dot{q}(t) = \Big(\tfrac{1}{4}p(t) - \tfrac{1}{2} - u(t)\Big)q(t), \tag{2}$$

$$y(t) = p(t) + q(t). \tag{3}$$

(a) As equilibrium solutions are constant solutions, their time derivatives are zero. Thus, considering (2) for $\dot{q} = 0$ leads to

$$0 = \big(\tfrac{1}{4}\bar{p} - \tfrac{1}{2} - 1\big)\bar{q}, \tag{4}$$

which has the solutions $\bar{q} = 0$ and

$$\bar{p} = 4\tfrac{3}{2} = 6. \tag{5}$$

As we are interested in an equilibrium for which both $\bar{p} > 0$ and $\bar{q} > 0$, the first solution is discarded. Then, considering (1) for $\bar{p} = 6$ yields

$$0 = 2 - \tfrac{6}{6} - \tfrac{1}{4}\bar{q} \quad \Rightarrow \quad \bar{q} = 4. \tag{6}$$

Thus, the desired equilibrium solution is

$$\begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}, \tag{7}$$

and note that the corresponding output reads $\bar{y} = 10$.

(b) For ease of notation, define

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}, \qquad \bar{x} = \begin{bmatrix} \bar{p} \\ \bar{q} \end{bmatrix}, \tag{8}$$

and let the function f be given by

$$f(x,u) = \begin{bmatrix} \big(2 - \tfrac{1}{6}x_1 - \tfrac{1}{4}x_2\big)x_1 \\ \big(\tfrac{1}{4}x_1 - \tfrac{1}{2} - u\big)x_2 \end{bmatrix}. \tag{9}$$

Then, the linearization of the dynamics (around the equilibrium $\bar{x}$ for constant input $\bar{u} = 1$) is given by

$$\dot{\tilde{x}}(t) = \frac{\partial f}{\partial x}(\bar{x},\bar{u})\tilde{x}(t) + \frac{\partial f}{\partial u}(\bar{x},\bar{u})\tilde{u}(t), \tag{10}$$

with the perturbations are defined as

$$\tilde{x} = x - \bar{x}, \qquad \tilde{u} = u - \bar{u}. \tag{11}$$

Computation of the partial derivatives leads to

$$\frac{\partial f}{\partial x}(x,u) = \begin{bmatrix} 2 - \tfrac{1}{3}x_1 - \tfrac{1}{4}x_2 & -\tfrac{1}{4}x_1 \\ \tfrac{1}{4}x_2 & \tfrac{1}{4}x_1 - \tfrac{1}{2} - u \end{bmatrix}, \qquad \frac{\partial f}{\partial u}(x,u) = \begin{bmatrix} 0 \\ -x_2 \end{bmatrix}, \tag{12}$$

such that evaluation at the equilibrium solution yields

$$A = \frac{\partial f}{\partial x}(\bar{x}, \bar{u}) = \begin{bmatrix} -1 & -1.5 \\ 1 & 0 \end{bmatrix}, \qquad B = \frac{\partial f}{\partial u}(\bar{x}, \bar{u}) = \begin{bmatrix} 0 \\ -4 \end{bmatrix}. \tag{13}$$

Note that the output equation is linear, which can be written in the perturbation coordinates as

$$\tilde{y} = C\tilde{x}, \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \tag{14}$$

where $\tilde{y} = y - \bar{y}$. Then, the final linearized reads

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\tilde{x}(t) + B\tilde{u}(t), \\ \tilde{y}(t) &= C\tilde{x}(t), \end{aligned} \tag{15}$$

with the matrices A and B as in (13) and C as in (14).

(c) Stability of the linearized system (15) is determined by the eigenvalues of the matrix A given in (13). In order to compute the eigenvalues, compute

$$\det\big(\lambda I - A\big) = \begin{vmatrix} \lambda + 1 & 1.5 \\ -1 & \lambda \end{vmatrix} = (\lambda + 1)\lambda + 1.5 = \lambda^2 + \lambda + 1.5. \tag{16}$$

Thus, the roots of the characteristic equation

$$\lambda^2 + \lambda + 1.5 = 0, \tag{17}$$

are sought. Note that stability can also be determined by applying the Routh-Hurwitz criterion to the polynomial (16). Solving (17) using the quadratic formula gives

$$\lambda = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1.5}}{2} = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5}i. \tag{18}$$

Note that the real parts of both roots are strictly negative, such that the linearized system is asymptotically stable.

Problem 2

Consider the polynomial

$$p(\lambda) = \lambda^4 + a\lambda^3 + 4\lambda^2 + 2a\lambda + b, \tag{19}$$

with a and b real numbers. To determine whether the polynomial (19) is stable, consider the following table:

	λ^4	λ^3	λ^2	λ^1	λ^0	
$a\times$	1	a	4	$2a$	b	
$1\times$	a		$2a$			
		a^2	$2a$	$2a^2$	ab	(step 1)
$2\times$		a	2	$2a$	b	(after dividing by a, note $a \neq 0$)
$a\times$		2		b		
$a(4-b)\times$			4	$a(4-b)$	$2b$	(step 2)
$4\times$			$a(4-b)$			
				$a^2(4-b)^2$	$2ab(4-b)$	(step 3)

On the basis of the table, the following conclusions can be drawn. Considering the initial polynomial, it is necessary for stability that all coefficients have the same sign. This immediately leads to the conditions $a > 0$ and $b > 0$.

Since $a > 0$, division by a after step 1 is allowed. Then, following the same reasoning for the polynomial obtained after step 2, it is necessary that $4 - b > 0$, i.e., $b < 4$.

Finally, the polynomial obtained after step 3 reads

$$a^2(4-b)^2\lambda + 2ab(4-b) = 0. \tag{20}$$

Since $a(4-b) > 0$ (as $a > 0$ and $0 < b < 4$), division by $a(4-b) > 0$ leads to

$$a(4-b)\lambda + 2b = 0 \quad \Rightarrow \quad \lambda = -\frac{2b}{a(4-b)}, \tag{21}$$

such that $\lambda < 0$ for all $a > 0$ and $0 < b < 4$, i.e., the polynomial obtained after step 3 is stable. Consequently, the polynomial (19) is stable for all parameters satisfying

$$0 < a, \qquad 0 < b < 4. \tag{22}$$

Problem 3

Consider the system

$$\dot{x} = \begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u. \tag{23}$$

(a) Controllability can be evaluated by computing the matrix

$$[\, B \;\; AB \,]. \tag{24}$$

Computing terms individually leads to

$$B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad AB = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \tag{25}$$

such that

$$[\, B \;\; AB \,] = \begin{bmatrix} -1 & 4 \\ 1 & 0 \end{bmatrix}. \tag{26}$$

It is readily seen that the rank of this matrix equals 2, such that (23) is controllable.

(b) Since (23) is controllable, there exists a nonsingular matrix T such that

$$T^{-1}AT = \begin{bmatrix} 0 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix}, \qquad T^{-1}B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

for some real numbers α_1 and α_2. In fact, this form is known as the controllable canonical form and the parameters α_i equal the coefficients (but with negative sign) of the characteristic polynomial of A. As such, these can be computed by considering

$$\det\big(\lambda I - A\big) = (\lambda+3)(\lambda-2) - 2 = \lambda^2 + \lambda - 8, \tag{27}$$

$$= \lambda^2 + a_1\lambda + a_2 \tag{28}$$

with $a_1 = 1$ and $a_2 = -8$. It then holds that

$$\alpha_1 = -a_2 = 8, \quad \alpha_2 = -a_1 = -1. \tag{29}$$

The corresponding transformation matrix T can be constructed by considering the vectors q_1 and q_2 as

$$q_2 = B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \tag{30}$$

$$q_1 = AB + a_1 B = AB + B = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \tag{31}$$

after which the matrix T can be constructed as

$$T = [\, q_1 \;\; q_2 \,] = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}. \tag{32}$$

Note that

$$T^{-1} = \tfrac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}, \tag{33}$$

after which it is easily checked that

$$T^{-1}AT = \tfrac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} = \tfrac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -8 & 4 \\ 8 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 8 & -1 \end{bmatrix}, \tag{34}$$

$$T^{-1}B = \tfrac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{35}$$

(c) In order to place the eigenvalues of $A + BF$ at -1 and -2, consider the polynomial

$$(\lambda + 1)(\lambda + 2) = \lambda^2 + 3\lambda + 2 = 0, \tag{36}$$

which has the desired eigenvalues as roots (and is monic). Define a matrix

$$\bar{F} = \begin{bmatrix} \bar{F}_1 & \bar{F}_1 \end{bmatrix} \tag{37}$$

and let $\bar{A} = T^{-1}AT$, $\bar{B} = T^{-1}B$ as in (34) and (35), respectively. Then,

$$\bar{A} + \bar{B}\bar{F} = \begin{bmatrix} 0 & 1 \\ 8 + \bar{F}_1 & -1 + \bar{F}_2 \end{bmatrix}, \tag{38}$$

which has the characteristic polynomial

$$\lambda^2 + (1 - \bar{F}_2)\lambda + (-8 - \bar{F}_1) = 0. \tag{39}$$

Matching coefficients of (39) and (36) leads to

$$\bar{F}_1 = -10, \qquad \bar{F}_2 = -2. \tag{40}$$

Next, noting that

$$T(\bar{A} + \bar{B}\bar{F})T^{-1} = T\bar{A}T^{-1} + T\bar{B}\bar{F}T^{-1} = A + B\bar{F}T^{-1}, \tag{41}$$

leads to the desired feedback F as

$$F = \bar{F}T^{-1} = \tfrac{1}{4}\begin{bmatrix} -10 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -4 \end{bmatrix}. \tag{42}$$

Problem 4

Consider the system

$$\dot{x} = \begin{bmatrix} -2 & 0 & 0 \\ -3 & 1 & 4 \\ -1 & 1 & -2 \end{bmatrix} x + \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x. \tag{43}$$

(a) Note that the system matrix has a block-triangular structure

$$A = \left[\begin{array}{c|cc} -2 & 0 & 0 \\ \hline -3 & 1 & 4 \\ -1 & 1 & -2 \end{array}\right], \tag{44}$$

of which the upper-left block is stable (i.e., it has eigenvalue -2). The eigenvalues of the lower-right can be obtained by solving

$$\begin{aligned} 0 = \det\left(\lambda I - A_{22}\right) &= \begin{vmatrix} \lambda - 1 & -4 \\ -1 & \lambda + 2 \end{vmatrix} \\ &= (\lambda - 1)(\lambda + 2) - 4 = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2), \end{aligned} \tag{45}$$

leading to the eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 2$. As $\lambda_2 > 0$ (more formally, $\mathrm{Re}(\lambda_2) > 0$), the system (43) is not (internally) stable.

(b) Controllability can be checked by computing

$$[\, B \;\; AB \;\; A^2B \,] = \begin{bmatrix} 4 & -8 & 16 \\ 0 & -16 & 0 \\ -1 & -2 & -4 \end{bmatrix}. \tag{46}$$

It is clear that the first and third column are linearly dependent, such that the rank of this matrix equals 2. Consequently (as $2 < 3$), the system is not controllable.

(c) The reachable subspace $\mathcal{W}$ is given by

$$\mathcal{W} = \mathrm{im}[\, B \;\; AB \;\; A^2B \,], \tag{47}$$

such that, using (46),

$$\mathcal{W} = \mathrm{im} \begin{bmatrix} 4 & -8 \\ 0 & -16 \\ -1 & -2 \end{bmatrix} = \mathrm{im} \begin{bmatrix} 4 & 0 \\ 0 & -16 \\ -1 & -4 \end{bmatrix} = \mathrm{im} \begin{bmatrix} 4 & 0 \\ 0 & 4 \\ -1 & 1 \end{bmatrix}. \tag{48}$$

(d) The system (43) is stabilizable if

$$\mathrm{rank}[\, \lambda I - A \;\; B \,] = 3 \tag{49}$$

for all λ such that $\mathrm{Re}(\lambda) > 0$ (i.e., for all unstable eigenvalues). As a result of (a), only the eigenvalue $\lambda_2 = 2$ is unstable, leading to

$$[\, \lambda_2 I - A \;\; B \,] = \begin{bmatrix} 4 & 0 & 0 & 4 \\ 3 & 1 & -4 & 0 \\ 1 & -1 & 4 & -1 \end{bmatrix}, \tag{50}$$

which can be seen to have rank 3. Consequently, the system is stabilizable.

(e) To evaluate whether (43) is observable, consider the matrix

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -3 & 1 & 4 \\ -1 & 5 & -4 \end{bmatrix}. \tag{51}$$

It can be seen that this matrix has rank 3, such that the system is observable.

(f) As the system is observable (see (e)), it is detectable.

Problem 5

Consider the linear system

$$\dot{x} = Ax + Bu,$$
$$y = Cx,$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$. Let $\mathcal{W}$ and $\mathcal{N}$ be the reachable subspace and unobservable subspace, respectively, and denote the impulse response as

$$h(t) = Ce^{At}B.$$

To show that $h(t) = 0$ for all $t \in \mathbb{R}$ if and only if $\mathcal{W} \subset \mathcal{N}$, necessity and sufficiency are considered separately. Note however that there are various other approaches for proving the result.

(if) Recall the definitions of $\mathcal{W}$ and $\mathcal{N}$ as

$$\mathcal{W} = \operatorname{im}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}, \qquad \mathcal{N} = \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}. \tag{52}$$

Then, the condition $\mathcal{W} \subset \mathcal{N}$ implies that

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0, \tag{53}$$

such that

$$CA^kB = 0 \tag{54}$$

for $k = 0, 1, \ldots, 2n-2$. By Cayley-Hamilton, the condition (54) holds in fact for all nonnegative integers k. Using the Taylor series expansion of the matrix exponential, it holds that

$$h(t) = Ce^{At}B = C\left(\sum_{k=0}^{\infty} \frac{t^k A^k}{k!}\right)B = \sum_{k=0}^{\infty} CA^kB\frac{t^k}{k!} = 0, \tag{55}$$

which finalizes the proof of this part.

(only if) To prove the converse, the Taylor series expansion (55) can be used to conclude (54) (for all nonnegative integer k), which in turn implies (53) and proves the result $\mathcal{W} \subset \mathcal{N}$.